Monte Carlo simulations of dissipative quantum Ising models

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The dynamical critical exponent z is a fundamental quantity in characterizing quantum criticality, and it is well-known that the presence of dissipation in a quantum model has significant impact on the value of z. Studying quantum Ising spin models using Monte Carlo methods, we estimate the dynamical critical exponent z and the correlation length exponent ν for different forms of dissipation. For a two-dimensional quantum Ising model with Ohmic site dissipation, we find $z \approx 2$ as for the corresponding one-dimensional case, whereas for a one-dimensional quantum Ising model with Ohmic bond dissipation, we obtain the estimate $z \approx 1$.

I. INTRODUCTION

Conventionally, quantum criticality can be described by a quantum-to-classical mapping,[1] whereby a d-dimensional quantum model is represented by a $(d+1)$-dimensional classical model in which the extra dimension corresponds to imaginary time, τ. It is well-known since the work of Hertz[2] that this temporal dimension and the spatial dimensions do not necessarily appear on an equal footing. In the presence of dissipative terms in the action, for instance, long-range interactions are introduced in the imaginary time direction,[3,4] making the model behave as if it were $(d+z)$-dimensional rather than $(d+1)$-dimensional. The dynamical critical exponent z can be regarded as a measure of the anisotropy between the temporal dimension and the spatial dimensions, as defined by the scaling of the temporal correlation length, $\xi_\tau \sim \xi^z$. Here, $\xi \sim |K - K_c|^{-\nu}$ is the spatial correlation length upon approaching a quantum critical point $K = K_c$, with K being some arbitrary (nonthermal) coupling constant. Knowing the value of z is therefore of fundamental importance in the study of quantum phase transitions, especially since this critical exponent determines the appearance of the quantum critical regime at finite temperatures above the quantum critical point.[5,6] Such quantum critical points with an accompanying quantum critical region have been suggested to be responsible, for instance, for the anomalous behavior of the normal phase of high-T_c cuprate superconductors.[7,8]

To illustrate the effect of dissipation on the dynamical critical exponent, consider first a generic ϕ^4-type nondissipative quantum field theory. The bare inverse propagator can be obtained from the quadratic part of the action as $\mathbf{q}^2 + \omega^2$, meaning that one has isotropic scaling between the spatial dimensions and the temporal dimension, i.e., $z = 1$. Adding local Ohmic dissipation by coupling each spin to a bath of harmonic oscillators,[3] the inverse propagator is modified to $\mathbf{q}^2 + \omega^2 + |\omega|$. Assuming a phase transition to an ordered state and taking the limit $\mathbf{q} \to \mathbf{0}$, $\omega \to 0$, the dissipative term $|\omega|$ will always dominate over the dynamic term ω^2, and so, by using $\omega \sim q^z$, we may naively make the prediction $z = 2$. Note that according to this argument, the dynamical critical exponent for a given action is independent of the spatial dimensionality of the system. We will refer to these scaling arguments as naive scaling, and postpone any discussion of caveats and other possible scaling choices to Sec. IV.

If one replaces this Ohmic site dissipation with dissipation that also couples in space and not just in time, this situation may change significantly. A common form of dissipation in the context of arrays of resistively shunted Josephson junctions and related models, is the Ohmic dissipation of gradients, i.e., of the bond variable that is the difference of the quantum phase between the superconducting elements.[9] In Fourier space, this bond dissipation corresponds to an inverse propagator $\mathbf{q}^2 + \omega^2 + \mathbf{q}^2 |\omega|$. (See, however, Sec. IV.) Once again letting $\mathbf{q} \to \mathbf{0}$, $\omega \to 0$, we can from naive scaling expect the dissipation to be weaker than in the onsite case since in this limit $\mathbf{q}^2 |\omega| \ll \mathbf{q}^2$ for any positive z. A possible value is therefore $z = 1$, for which the spatial term balances the dynamic term and dissipation can be considered perturbatively irrelevant in renormalization group sense.

Simple arguments of the kind given above have been the approach most commonly used whenever a dynamical critical exponent is to be determined. In recent years there has however been progress toward calculating the corrections to these lowest-order estimates for z both by field-theoretical renormalization group methods[10–12] and by Monte Carlo methods.[12–15] In addition, there has also been considerable recent interest in dissipative systems exhibiting more exotic forms of quantum criticality where the critical exponents are varying continuously.[16–18]

The most notable advance from our point of view is, however, the work by Werner *et al.*[13] justifying numerically the naive scaling estimate for the Ising spin chain with site dissipation by extensive Monte Carlo simulations. More precisely, it was found that the dynamical critical exponent was universal and satisfied $z = 2 - \eta$, with an anomalous scaling dimension $\eta \approx 0.015$. Apart from Ref. 13, almost no Monte Carlo simulations have been performed on extended quantum dissipative models. (See, however, Refs. 15 and 19 for reviews of Monte Carlo simulation for dissipative systems and quantum phase transitions.) The present work can therefore be regarded as a natural extension of the work done by Werner *et al.*, but more importantly as a first step toward more complex dissipative quantum models with bond dissipation. For instance, the dissipative XY model with bond dissipation is very interesting both as a model of granular superconductors or other systems which may be modeled as Josephson junction arrays.[9] In particular, such a dissipative XY model[20] and related Ashkin-Teller models[21,22] have been proposed to describe quantum critical fluctuations of loop-current order in cuprate superconductors.

Finding a value of z is also of considerable interest for purely classical models that include strongly anisotropic interactions. The reason is simply that performing a finite-size analysis to find the critical coupling or critical exponents requires a choice of system sizes that reflects an anisotropy in the scaling of the correlation lengths. In other words, one ideally needs to know the relative correlation length exponent $\nu_\tau/\nu = z$ *a priori* for the finite-size analysis to be correct.

In this work, we seek to employ Monte Carlo simulations of Ising models to answer the following questions: (1) can we confirm numerically that the dynamical critical exponent is indeed independent of dimensionality? (Neglecting the assumed small η) (2) how will the dynamical critical exponent for Ising variables change if one replaces the site dissipation with dissipation that also acts in space? The first question will be addressed in Sec. II, where we study the two-dimensional (2D) quantum Ising model with site dissipation. In Sec. III, we turn to the second question by studying a one-dimensional (1D) quantum Ising chain with bond dissipation in a similar manner. The results will be related to the naive scaling arguments for z, after which we conclude in Sec. V.

II. 2D QUANTUM ISING MODEL WITH SITE DISSIPATION

We first consider a quantum Ising spin model in two spatial dimensions coupled to a bath of harmonic oscillators,[3] i.e., a higher-dimensional version of the model considered in Ref. 13. In Fourier space, the quadratic part of the action for this model can be written as

$$S = \sum_{\mathbf{q}} \sum_{\omega} \left(\tilde{K}\mathbf{q}^2 + \tilde{K}_\tau \omega^2 + \frac{\alpha}{2}|\omega| \right) \sigma_{\mathbf{q},\omega} \sigma_{-\mathbf{q},-\omega}, \qquad (1)$$

where σ is the Ising field. The discretized real space representation on a $L \times L \times L_\tau$-lattice then reads

$$\begin{aligned} S = -K \sum_{x=1}^{L} \sum_{y=1}^{L} \sum_{\tau=1}^{L_\tau} [\sigma_{x,y,\tau}\sigma_{x+1,y,\tau} + \sigma_{x,y,\tau}\sigma_{x,y+1,\tau}] \\ -K_\tau \sum_{x=1}^{L} \sum_{y=1}^{L} \sum_{\tau=1}^{L_\tau} \sigma_{x,y,\tau}\sigma_{x,y,\tau+1} \\ + \frac{\alpha}{4} \sum_{x=1}^{L} \sum_{y=1}^{L} \sum_{\tau \neq \tau'}^{L_\tau} \left(\frac{\pi}{L_\tau} \right)^2 \frac{(\sigma_{x,y,\tau} - \sigma_{x,y,\tau'})^2}{\sin^2(\pi/L_\tau|\tau - \tau'|)}. \end{aligned} \qquad (2)$$

We have assumed a spatially isotropic system, so that $K_x = K_y = K$. Periodic boundary conditions are implicit in the imaginary time direction and are also applied for the spatial directions. Note that our representation is equivalent to that of Ref. 13, although superficially appearing slightly different.

We could, as Werner *et al.*, take a quantum Ising model in a transverse magnetic field as a starting point, and the field would then give rise to the quantum dynamics of the spins as represented by the second line in the action in Eq. (2). However, in this work we are not interested in the effect of a transverse field *per se*, and will therefore treat the dynamic term as a phenomenological term of unspecified origin. (See,
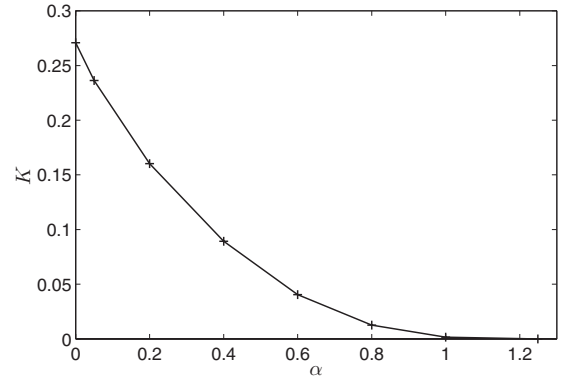

FIG. 1. Phase diagram for the 2D quantum Ising model with site dissipation for $K_\tau = -1/2 \ln(\tanh 1)$. The ordered phase is found for large values of spatial coupling K and dissipation strength α.

however, Sec. IV). In the following, we will fix the value of the dynamic coupling of the Ising field to $K_\tau = -1/2 \ln(\tanh 1) \approx 0.1362$ and vary the spatial coupling K. For the $(1+1)$-dimensional model,[13] this choice ensures that $K_c = 1$ for $\alpha = 0$, whereas in the $d = 2$ case it is chosen primarily for computational convenience, and to allow for direct comparison with the $d = 1$ case. For the Monte Carlo simulations, we have used an extension of the Wolff cluster algorithm[23] by Luijten and Blöte,[24] which very effectively treats the long-range interaction in the imaginary time direction. We have mainly used an implementation of the Mersenne Twister[25] random number generator (RNG), but also confirmed that other RNGs yielded consistent result. We also make use of Ferrenberg-Swendsen[26] reweighting techniques which enable us to vary K continuously after the simulations have been performed.

We will first present the phase diagram for this model in the α-K plane, as shown in Fig. 1. The phase diagram for the $(2+1)$-dimensional model is very similar to that for its $(1+1)$-dimensional counterpart, with a disorder-order phase transition for increasing dissipation and/or spatial coupling. Along the α-axis, a temporally ordered state is reached at $\alpha = \alpha_c$ through a purely dissipative phase transition when $K = 0$, in which case the model is simply a collection of decoupled $(0+1)$-dimensional dissipative two-level systems. The long-range interaction in the temporal chains decays as $1/|\tau_i - \tau_j|^2$, accordingly, the phase transition is of a kind closely related to the Kosterlitz-Thouless transition,[27] in which the ordered phase consists of tightly bound kinks and antikinks.

With the same temporal coupling values as for the $d = 1$ case, we can with relative ease determine the critical dissipation strength α_c for the independent subsystems, see the result stated in Ref. 13.

We have chosen a somewhat more quantitative approach to determine the dynamical critical exponent z than the one given in the presentation of Werner *et al.*, so we will use the exposition of our results to detail the method. This method is essentially the same as the one applied by the authors of Refs. 28 and 29 for spin glasses in a transverse field, but as it is rather scantily described in the literature, we include it here for completeness.

The basis of our approach is as follows. For systems with isotropic scaling, a well-known method to determine the

value of the critical coupling is to calculate the Binder ratio

$$Q = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}, \tag{3}$$

and use this to plot the Binder cumulant $g \equiv 1 - Q/3$ as a function of coupling for several (e.g., cubic, in the $(2+1)$-dimensional case) system sizes. The Binder cumulant at the critical coupling is independent of system size (to leading order in L), and the crossing point of $g(K)$ for two different system sizes thus defines the (pseudo)critical point.

However, this finite-size scaling approach breaks down when the system size scales anisotropically. In this case, the scaling at criticality is given as a function with two independent scaling variables instead of just one,

$$Q(L, L_\tau) = \mathcal{G}\left(\frac{L}{\xi}, \frac{L_\tau}{\xi_\tau}\right), \tag{4}$$

and anisotropic systems according to $L_\tau \propto L^z$ are the appropriate choice instead of cubic systems. Hence, given the value of z, one should also observe data collapse as a function of L_τ/L^z for the Binder cumulant curves at the critical point.

In order to find z self-consistently, we consider first the Binder cumulant as a function of L_τ for given α, K and L. For very small L_τ, the system appears effectively two-dimensional, and consequently the increased influence of fluctuations makes this system more disordered than the corresponding three-dimensional system. In the opposite limit of $L_\tau \to \infty$ the system appears effectively one-dimensional, and with $L_\tau \gg \xi_\tau$ the system is again disordered. As g is a measure of the degree of order in the system, $g \to 0$ in both the above limits, and accordingly g must have a maximum for some finite value $L_\tau = L_\tau^*$. One way of interpreting L_τ^* is as the temporal size for which the system appears as isotropic as it possibly can be (or optimally three-dimensional), the anisotropic interactions taken into account.

The details of our procedure are as follows. First, we sample the Binder ratio as a function of coupling K for a large number of system sizes. For each value of L, we choose at least 14 values of L_τ close to the presumed peak position L_τ^* for the extent of the imaginary time dimension. The procedure for estimating z then follows in three steps. For each K, curves of the Binder cumulant g for all L are plotted as a function of L_τ, corresponding to the plot shown in panel (a) of Fig. 2. Second, a fourth order polynomial fit is made to these curves, localizing the points (L_τ^*, g^*) defining the peaks of the functions $g(L_\tau)$ with good precision. The obtained values for the peak Binder cumulants for each L are then plotted as a function of K, as shown in panel (b) of Fig. 2. A value for the critical coupling K_c can be found by estimating the value K to which the crossing point for two subsequent values of L converges for $1/L \to 0$. The third step for finding the dynamical critical exponent is a simple finite-size scaling analysis of the peak positions L_τ^* of the curves $g(L_\tau)$ as shown in panel (c) of Fig. 2, assuming the relation $L_\tau^* = aL^z$, with a being a nonuniversal prefactor. Finally, one may
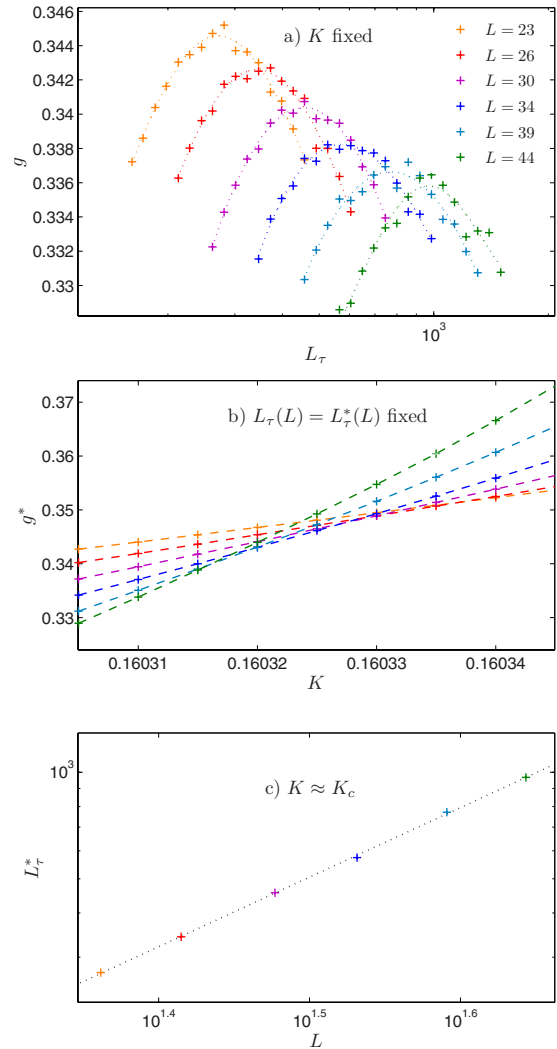

FIG. 2. (Color online) Illustration of the procedure for estimating the dynamical critical exponent z, as described in the text, here for the 2D quantum Ising model with site dissipation and $\alpha = 0.2$. (a) The Binder cumulant g as a function of temporal system size L_τ for a number of spatial system sizes L at $K = 0.160312$. (b) The peak value Binder cumulant g^* as a function of coupling K. (c) Finite-size analysis of the peak position of L_τ^* as a function of spatial system size L at criticality, $K_c = 0.160312(2)$, which yields the estimate $z = 1.97(3)$. The straight line represents a least-squares fit to the data points.

check the self-consistency of the obtained values for K_c and z by plotting the putative data collapse of the Binder cumulant as a function of L_τ/L^z, cf. Eq. (4).

Before moving on, we comment on the two interrelated (subleading) finite-size effects in the crossing point of Fig. 2: the crossing point between two subsequent Binder curves moves toward lower coupling for increasing system size, and accordingly the Binder cumulant at the crossing point decreases for increasing L. Consequently, the value of $g^*(K = K_c)$ will never be independent of system size L for finite systems. However, in our experience, this vertical deviation from collapse of the Binder curves—which is particularly evident when focusing on the peak of the Binder curves as in our analysis—does not itself affect the finite-size
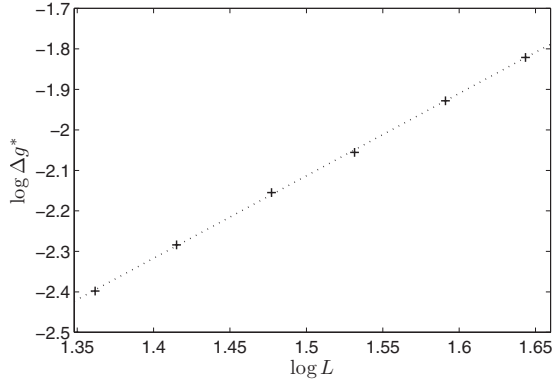
FIG. 3. Finite-size analysis for obtaining $1/\nu$ for the 2D quantum Ising model. Here we have evaluated the slope of the Binder cumulant g^* around $K=0.160312$ for $\alpha=0.2$, which yields $\nu=0.49(1)$. The straight line represents a least-squares fit to the data points.

estimate for z. More important is a possible horizontal deviation. Likewise, a slow convergence of the crossing points to K_c complicates the determination of the critical coupling for finite systems. The resulting uncertainty in z is dominated by this uncertainty in K_c, at least for the $d=2$ case.

It might be possible to obtain better precision for the critical coupling by using the finite-size analysis technique presented in Ref. 30 for the crossing points, but in the present case with an additional (and unknown) finite-size effect in z, this more rigorous approach seems by no means straightforward. To ensure that finite-size effects are negligible, we have checked the dependence of z on the lowest value of L included in the fitting procedure. In the analysis illustrated in Fig. 2, we have only retained system sizes such that the value of z seems to have converged. For the case $\alpha=0.2$ considered above, the resulting estimate is $z=1.97(3)$. No significant variation in the dynamical critical exponent is observed for stronger dissipation, and we conclude that we have $z \approx 2$ along the critical line. However, we have not been able to determine conclusively whether or not one has exactly anomalous scaling dimension $\eta=0$ in the relation $z=2-\eta$, which might be expected[10] since the value $d+z$ lies at the upper critical dimension for this phase transition for $d=2$.

We also give an estimate of the correlation length exponent ν using the peak values $g^*(K)$ of the Binder cumulant. The leading order scaling properties of the Binder ratio can be stated as[31] $Q(K,L)=\widetilde{\mathcal{G}}([K-K_c]L^{1/\nu})$, and assuming negligible finite-size effects in the obtained dimensions $L_\tau^*(L)$, one finds the finite-size relation

$$\log \frac{dg^*}{dK} = C + \frac{1}{\nu}\log L, \qquad (5)$$

The slope dg^*/dK is estimated by the finite difference Δg^* over a small coupling interval around K_c, and C is an unimportant constant. The resulting finite-size analysis for $\alpha=0.2$ is illustrated in Fig. 3, and we find $\nu=0.49(1)$. This is very close to the expected (mean-field) value $\nu=1/2$ (Ref. 10), which is reasonable given that $z \approx 2$.

We finally note that, whereas increasing α does not lead to a significant change of z, it certainly does increase the prefactor a of the scaling relation $L_\tau \sim L^z$ and thereby the peak position L_τ^*. This reflects the increased anisotropy of the interactions, and can be seen also for $\alpha=0$ when K and K_τ are allowed to vary freely. At criticality one has $a=1$ for $K_\tau=K$, with increasing a for increasing anisotropy K_τ/K. In fact, for the analytically solvable 2D Ising model there even exists an exact mapping between system size anisotropy (i.e., a) and interaction anisotropy (i.e., K_τ/K).[32]

III. QUANTUM ISING CHAIN WITH BOND DISSIPATION

In this section, we will consider a $(1+1)$-dimensional quantum Ising model where the dissipative quantities of interest are bond variables involving Ising spins, rather than individual Ising spins themselves. The specific form of this dissipation kernel has been proposed as a candidate for describing the origin of the anomalous normal state properties of the cuprate high-T_c superconductors,[20] but in that case involving two sets of Ising spin on each lattice point. Such a model, unlike the one we will consider, may be mapped onto a four-state clock model, and may be approximated by an XY model with a fourfold symmetry breaking field, which in the classical case in two spatial dimensions is perturbatively irrelevant near criticality on the disordered side. Due to the degrees of freedom in our model being Ising spins with a spin gap, the present model should therefore not be regarded as directly comparable to a dissipative XY model that the authors of Ref. 20 consider. It should rather be regarded as a simple, but spatially extended model system, illustrating how bond dissipation can affect a quantum phase transition, which is certainly an important question on its own right.

In Fourier space, the action is given by

$$S = \sum_{\mathbf{q}} \sum_{\omega} \left(\widetilde{K}\mathbf{q}^2 + \widetilde{K}_\tau\omega^2 + \frac{\alpha}{2}|\omega||\mathbf{q}|^2 \right)\sigma_{\mathbf{q},\omega}\sigma_{-\mathbf{q},-\omega}. \qquad (6)$$

The real space representation of this system is given by the action

$$S = -K\sum_{x=1}^{L}\sum_{\tau=1}^{L_\tau} \sigma_{x,\tau}\sigma_{x+1,\tau} + K_\tau\sum_{x=1}^{L}\sum_{\tau}^{L_\tau} \sigma_{x,\tau}\sigma_{x,\tau+1}$$
$$+ \frac{\alpha}{2}\sum_{x=1}^{L}\sum_{\tau\neq\tau'}^{L_\tau} \left(\frac{\pi}{L_\tau}\right)\frac{(\Delta\sigma_{x,\tau} - \Delta\sigma_{x,\tau'})^2}{\sin^2(\pi/L_\tau|\tau-\tau'|)}, \qquad (7)$$

cf. the site dissipation case in Eq. (2). Here, $\Delta\sigma_{x,\tau} \equiv \sigma_{x+1,\tau} - \sigma_{x,\tau}$

The interpretation of this representation remains mostly the same as in the previous section. The only difference is that the coupling to the heat bath is given in terms of the Ising field gradients rather than the Ising fields themselves. In the limit $\mathbf{q}\to 0$, $\omega\to 0$, we may anticipate from the Fourier representation of the action that the last term becomes irrelevant, which implies the value $z=1$ for the dynamical critical exponent. It is also evident from Eq. (7) that the bond dissipation is less effective than site dissipation in reducing quantum fluctuations. While site dissipation tends to align all
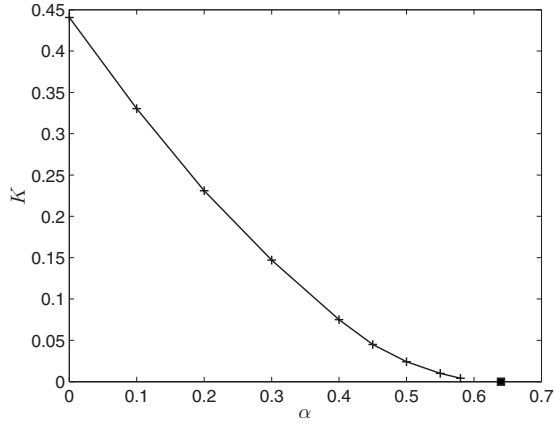
FIG. 4. Phase diagram of the quantum Ising chain with bond dissipation for $K_\tau = \ln(1+\sqrt{2})/2$. The ordered phase is found for large values of spatial coupling K and dissipation strength α. The filled square on the α axis represents an upper bracket for critical coupling α_c when the spatial coupling is tuned to zero, see the text.

spins in the temporal direction, the bond dissipation tends to align the *difference* of nearest-neighbor spins along the Trotter slices. At least in the presence of a finite coupling $K \neq 0$, this is a less effective way of reducing temporal fluctuations of individual spins than onsite dissipation.

When expanding the dissipative term, it becomes clear that it contributes to both ferromagnetic and antiferromagnetic long-range interaction. This renders the system intractable to the Luijten-Blöte variant cluster algorithm used in the previous section. This algorithm builds up clusters with sizes comparable to the entire system and flips these as a whole, resulting in extreme correlations.[33] No cluster algorithm that effectively handles competing interactions has come to the authors' attention.

In the Monte Carlo simulations, we have therefore used a parallel tempering[34,35] algorithm which adequately handles the critical slowing down in the critical regime. A number of independent systems perform random walks in the space of coupling values, and this enables the systems to effectively explore a rugged energy landscape like the one generated by the dissipation term in Eq. (7).

The coupling values are distributed according to the iteration procedure introduced by Hukushima,[36] which renders the accept ratio of the attempted exchange of two adjacent coupling values independent of the coupling value. Consequently, the systems are allowed to wander relatively freely through the space of coupling values, although even more sophisticated distribution algorithms are available in that respect.[35]

The parameter K_τ is fixed at $\ln(1+\sqrt{2})/2 \approx 0.4407$, the critical coupling K_c is thus the same as for the isotropic 2D Ising model when the dissipation strength is tuned to zero. Anticipating $z=1$, this choice also ensures that the simulations will be performed for convenient values of L and L_τ. The further steps necessary to find information about the critical properties are the same as discussed in Sec. II. The phase diagram of the system in the α-K plane is shown in Fig. 4.

For this model, the critical exponents are extracted for the two dissipation strengths $\alpha = 0.1$ and 0.2. In Fig. 5, we show
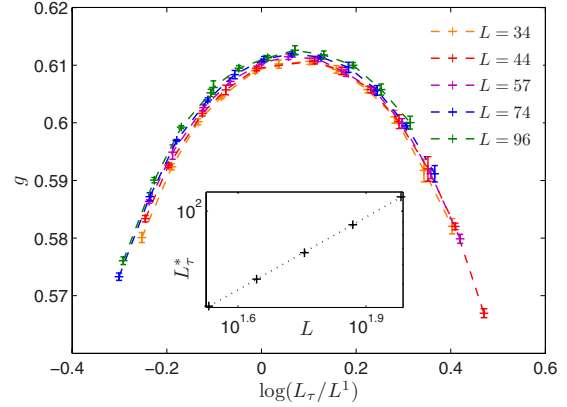

FIG. 5. (Color online) Data collapse of the Binder cumulant for $z=1$ at $K_c=0.3306$ for the 1D quantum Ising chain with bond dissipation and $\alpha=0.1$. The error bars are obtained from a jackknife analysis in the reweighting procedure. *Inset:* finite-size analysis resulting in dynamical critical exponent $z=1.007(15)$. The straight line represents a least-squares fit to the data points.

the results for the dynamical critical exponent for $\alpha=0.1$ as illustrated by the collapse of the Binder cumulant curves discussed in Sec. II for the value $z=1$. The results confirm the proposed value of z based in naive scaling arguments, and it appears that the bond dissipation term is indeed irrelevant.

The value of the dynamical critical exponent is very sensitive to finite-size effects and therefore challenging to obtain with the algorithm we have used given the limitations this entails. Increasing the dissipation strength makes these challenges more apparent, so to illustrate the dependence of z on system size we plot in Fig. 6 z as a function of system size for a fixed $K=K_c$ for $\alpha=0.2$. Note that three adjacent L values have been used to calculate every value for z, $\langle L \rangle$ denoting the average of these. The evolution of z is clearly seen to approach $z \approx 1$ in the thermodynamic limit. Even larger dissipation strengths tend to require much larger system sizes not practically feasible with the current algorithm. Results for such dissipation strengths are therefore not included here.

We have also attempted to extract the correlation length exponent ν for both dissipation strengths. When discarding
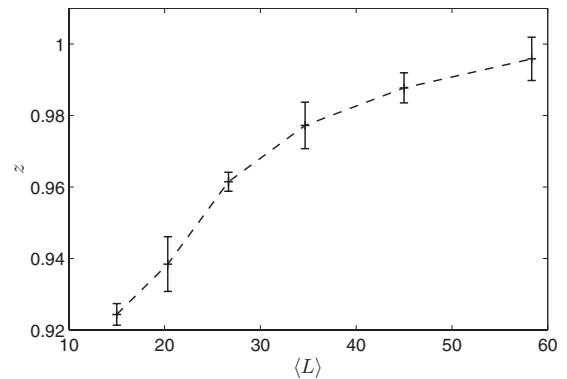

FIG. 6. The evolution of z as a function of system size for the 1D quantum Ising chain with bond dissipation with $\alpha=0.2$. Every point is calculated for the same coupling value $K_c=0.231$ from three adjacent system sizes, and the error bars are obtained from the least-squares fit in the finite-size analysis.
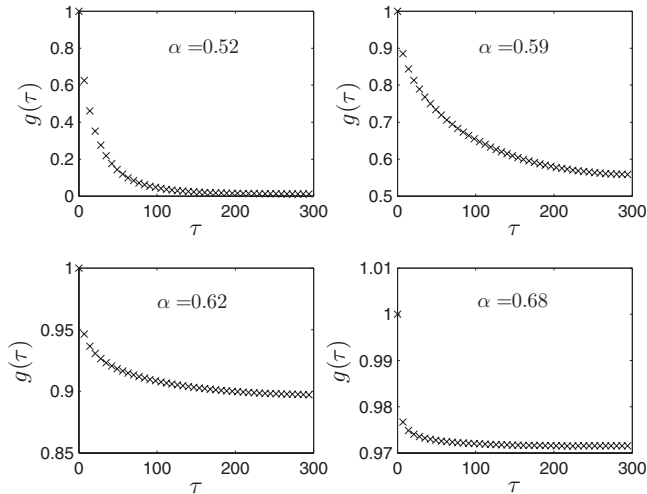
FIG. 7. The temporal spin-spin correlation function $g(\tau)$ $=\langle\sigma_{x,\tau}\sigma_{x,0}\rangle$ for the 1D quantum Ising chain with bond dissipation at $K=0$ with $L=20$ and $L_\tau=600$. The decay of the correlation function is illustrated for four different dissipation strengths as the system goes from the disordered phase ($\alpha=0.52$) to the ordered phase ($\alpha=0.68$).

FIG. 8. (Color online) The temporal correlation length ξ_τ as a function of dissipation strength α for the 1D quantum Ising chain with bond dissipation at $K=0$. Because of the extremely anisotropic scaling in this limit, and to be sure to avoid any spatial finite-size effects, we have chosen to fix $L=20$.

the smallest system sizes where finite-size effects are expected to be important, the values are found to be ν $=1.00(2)$ for $\alpha=0.1$ and $\nu=1.005(8)$ for $\alpha=0.2$. This corresponds well with the exact value $\nu=1$ expected for the universality class of a 2D Ising model.

Sufficiently strong dissipation brings the critical coupling K_c toward zero, and, as indicated on the α-axis of the phase diagram in Fig. 4, the model undergoes a purely dissipative phase transition at some critical dissipation strength α_c. The ground state at $K=0$ consists of columns in the direction of imaginary time of ordered Ising spins. However, the direction of ordering is in general not uniform, as can be seen from Eq. (7), since a column can be flipped as a whole with no cost of energy. This nonuniform order prohibits the use of Binder cumulant curves to determine the critical coupling, so the exact value of α_c is difficult to deduct from the simulations. These obstacles make an estimate of the dynamical critical exponent unfeasible by our methods. Furthermore, since this phase transition is not of Kosterlitz-Thouless nature, any variety of the method of Ref. 27 also seem to be inapplicable to this model.

To corroborate that, there is in fact a phase transition to an ordered state for increasing α also at $K=0$, we present in Fig. 7 results for the temporal spin-spin-correlation $g(\tau)$ $=\langle\sigma_{x,\tau}\sigma_{x,0}\rangle$. It is clear that this correlation function decays exponentially to zero for low dissipation strengths, while in the opposite limit of strong dissipation the correlation function quickly decays to some finite value. The character of the correlation function as α is tuned through the intermediate region is better illustrated in Fig. 8, where we have extracted the temporal correlation length ξ_τ. The diverging correlation length signifies a critical region with algebraic decay of the correlation function. The spatial correlation length ξ, on the other hand, we have found to be vanishing also in the critical region, and the behavior of the system depends only very weakly on its spatial extent L. From a crude finite-size analy-
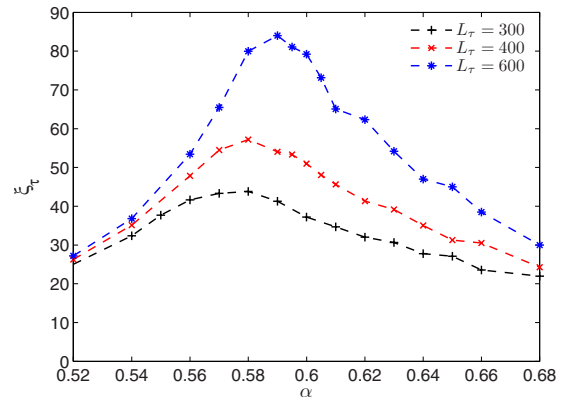
sis based on Fig. 8, we obtain the value $\alpha_c \lesssim 0.64$ as a best estimate for a upper bracket of the critical coupling, as we indicated in the phase diagram in Fig. 4.

IV. DISCUSSION

We will begin the discussion of our results by taking a closer look at the scaling arguments presented in Sec. I for finding the dynamical critical exponent. As indicated here, one important caveat of such arguments is that they only tell what exponent is naively expected to the lowest order approximation, and in general field-theoretical methods (see, e.g., Ref. 10) are needed to ascertain how higher order corrections modify this estimate. Furthermore, with several terms in the quadratic part of the action, it is not always obvious which terms should be required to balance at the critical point, or for which phase transitions this is valid.

For site dissipation, one obtains $z=2$ by balancing the spatial term and the dissipative term, since the dynamic term will be subdominant to the dissipative term for all positive z. For the bond dissipation case, a similar argument excludes the possibility $z=2$ for which the dissipative term would balance the dynamic term, since they both would be subdominant to the spatial term for all $z>1$. It is therefore interesting to ask if the possibility $z=0$, or alternatively $z\ll1$, can be considered. In the limit that z is strictly zero, a dissipative term on the form $|\omega|$ would balance the dynamic term whereas a dissipative term on the form $\mathbf{q}^2|\omega|$ would balance the spatial term, but in the latter case both would be subdominant to the dynamic term. One interpretation is that z $=0$ in both cases would imply unrestrained quantum fluctuations resulting in spatial correlations being infinitely stronger than temporal correlations, so that each Trotter slice is essentially independent. In this interpretation, a strictly vanishing dynamical critical exponent may however be considered unphysical since we are assuming a transition to uniform order for the entire $(d+1)$-dimensional system by taking the limit $\mathbf{q}\to\mathbf{0}$, $\omega\to0$.

Likewise, tuning $K_\tau\to0$ may be considered unphysical since one removes the origin of the quantum nature of the

system. For this reason one cannot say that there will exist a quantum phase transition with $z=0$ for the bond dissipation model even if the ω^2 term had been removed from the action. The origin of the ω^2 term in a physical quantum model can be a transverse magnetic field in the Ising case or a Josephson charging energy in the XY case, and the interpretation of the prefactor K_τ is in general the inertia of the degrees of freedom. Even though we have chosen to operate with a nonspecific parameter K_τ, we therefore do not regard taking $K_\tau=0$ admissible in our simulations.

The opposite limit of $z=\infty$ may similarly be interpreted as spatially local criticality with correlations in the imaginary time direction independent of (the vanishing) correlations in the spatial directions, see, e.g., Refs. 16, 17, and 20. This is trivially the case in the limit $K=0$ for site dissipation with $\alpha>\alpha_c$, although one may argue that z is undefined in that case as the system is strictly decoupled in the spatial directions. The same argument cannot be applied to bond dissipation. For that model, the system does not experience dimensional reduction as $K\to 0$, but is still dependent (although very weakly) on the spatial extent of the $(d+1)$-dimensional system. We should however note that the approach taken here for determining the dynamical critical exponent is not applicable when z is either strictly zero or infinite, and also for a constant value $z\gg 1$ it would be very difficult to determine the dynamical critical exponent for practically attainable lattice sizes. If, on the other hand, one has $z\to\infty$ in the sense of activated dynamical scaling, the method is in principle feasible.[19]

Before continuing the discussion of the bond dissipation, we comment further on the relation between the real space representation of $\mathbf{q}^2|\omega|$ and the form of the bond dissipation used in Eq. (7). When Fourier transforming $\mathbf{q}^2|\omega|\sigma_{\mathbf{q},\omega}\sigma_{-\mathbf{q},-\omega}$ from Eq. (6) and discretizing the resulting differential operators, we arrive at

$$S_{\mathbf{q}^2|\omega|} \propto -\left(\frac{\pi}{L_\tau}\right)^2 \frac{\Delta\sigma_{x,\tau}\cdot\Delta\sigma_{x,\tau'}}{\sin^2(\pi/L_\tau|\tau-\tau'|)}. \quad (8)$$

Now, writing out the last term of Eq. (7) and comparing with Eq. (8) shows that the Fourier space representation of the bond dissipation can be written as

$$S_{\text{bond}} = (\mathbf{q}^2|\omega| + C'\mathbf{q}^2)\sigma_{\mathbf{q},\omega}\sigma_{-\mathbf{q},-\omega}. \quad (9)$$

Here, C' depends weakly on dimensions for finite systems. In other words, the bond dissipation is of the same form as $\mathbf{q}^2|\omega|$ dissipation, but with renormalized spatial nearest-neighbor coupling, which however does not alter the critical exponents of the model. This extra term originates with the counterterm introduced to cancel out the renormalization of the bare potential that arises due to the coupling with a heat bath.[3] For the Ising model, this renormalization is responsible for stabilizing ferromagnetic order at $K>0$.

We will now turn to the analysis of simulations on finite lattices, in particular with respect to the scaling relation $L_\tau=aL^z$ and the system anisotropy expressed by it. To interpret our results, it is useful to consider the dependence of both z and a on the dissipation strength α, and the variation in these quantities can be understood as follows. If the dissipation

term is relevant and thus determining the universality class, we may assume that the value of z will be given by the form of this term even for infinitesimal $\alpha>0$ in the thermodynamic limit. In this case, increasing the dissipation strength α further will therefore not change z, but the prefactor a will have to change to reflect the increased interaction anisotropy. Correspondingly, when the dissipation term is an irrelevant perturbation, the dynamical critical exponent will remain $z=1$ in the thermodynamic limit. Upon increasing α, the dissipation will never grow strong enough to alter the universality class, but the nonuniversal prefactor a will in general change also in this case, and whether it increases or decreases is determined by how the dissipation changes the overall interaction anisotropy.

Regarding the evolution of a upon increasing α for the bond dissipation case, there are now two effects that must be considered separately. One implicit effect is that increasing α decreases $K=K_c$ at criticality, thereby increasing the anisotropy ratio K_τ/K, which results in a much larger a for large values of α. The other effect is that arising explicitly from the dissipation term and its contribution to the effective coupling strength in the imaginary time direction. Whereas a site dissipation term obviously increases the anisotropy when increasing the dissipation strength while keeping the other coupling values fixed, such an enhancement of a does not appear for bond dissipation. This can be seen—as we have checked—by evaluating a for increasing α for isotropic short-range coupling, i.e., $K_\tau=K$. One possible interpretation of this result is that although bond dissipation does not change universality, it favors $z<1$ behavior, which can also be recognized from Fig. 6. In other words, the dissipation term contributes to making the temporal dimension less ordered than the spatial dimension, in strong contrast to the case of site dissipation. This would in part explain why one needs much longer simulations and larger systems to obtain reliable results for strong bond dissipation.

Given that the exceedingly strong finite-size effects thwart a precise determination of z for higher values of α, one should in general also consider the possibility of continuously varying critical exponents. However, we have shown that $z\approx 1$ for $\alpha=0.1$ and presented solid arguments favoring that this is the case also for $\alpha=0.2$, as it is obviously also in the limit $\alpha=0$. Therefore, if the exponents are in fact continuously varying, they begin to vary only for dissipation strengths above $\alpha>0.2$, and would furthermore have to be varying very slowly.

V. CONCLUSION

This work represents a further step toward simulations of physically interesting extended quantum systems with dissipation. Using Monte Carlo methods, we have studied a model similar to that by Werner *et al.*,[13] but with higher spatial dimensionality, as well as a model with one spatial dimension but with bond dissipation instead of site dissipation. We have found that the $(2+1)$-dimensional model with site dissipation has a dynamical critical exponent very close to the corresponding $d=1$ model, i.e., $z\approx 2$. Bond

dissipation, on the other hand, is fundamentally different, and our results strongly suggest that this form of dissipation is irrelevant to the universality class, i.e., $z \approx 1$ and nonvarying. We therefore believe that the same dynamical critical exponent also applies to $(2+1)$-dimensional models with bond dissipation for the same degrees of freedom, although we have not been able to reach sufficiently large systems to show this convincingly by numerical means. In both cases, the numerical estimates for the dynamical critical exponent is consistent with those found by naive scaling arguments on the quadratic part of the action.

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